METHOD OF ELASTIC SOLUTIONS IN THEORY OF PLASTIC FLOW

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In flow theory the "step" method [1] is generally used to solve problems of complex loading. The loading process is divided into steps, within which the differential relations are replaced by finite-difference relations.

Below, a method, completely analogous to the method of elastic solutions in the theory of small elasto-plastic deformations [2], is proposed for the solution of problems in flow theory [1], on the assumption that there is no relaxation of load at any point in the body.

Consider a body made of an incompressible material obeying the law

$$2Gd\varepsilon_{ij} = dS_{ij} + dF(T)\,S_{ij} \qquad (dT \geqslant 0)\ . \tag{1}$$

The following notation will be used: G is the modulus of rigidity, S_{ij} is the stress deviator, ε_{ij} is the deformation tensor, and T is the stress intensity:

$$T = \sqrt{3/2}\,\sqrt{S_{ij}S_{ij}} \tag{2}$$

The function $F(T)$ can, for example, be found from a simple tensile test and, because it is determined only for positive T, it can always be represented in the form

$$F(T) = \sum_{\alpha=1} A_{2\alpha}' T^{2\alpha} = \sum_{\alpha=1} A_{2\alpha} (S_{ij}S_{ij})^{\alpha}. \tag{3}$$

For simple tension we get

$$3G\varepsilon_x = \sigma_x + \sum_{\alpha=1} B_{2\alpha+1}\sigma_x^{2\alpha+1}, \qquad B_{2\alpha+1} = \frac{2\alpha}{2\alpha+1} A_{2\alpha}', \quad A_{2\alpha}' = \frac{4\alpha}{3(2\alpha+1)} A_{2\alpha}\ . \tag{4}$$

Let the surface F_i and mass V_i forces vary with increase in the load parameter λ (which may be time) in such a way that there is no relaxation of load at any point in the body ($dT > 0$) and F_i and V_i can be written in the form

$$F_i = \sum_{k=1} F_i^{(k)} \lambda^k\ , \qquad V_i = \sum_{k=1} V_i^{(k)} \lambda^k. \tag{5}$$

where $F_i^{(k)}$ and $V_i^{(k)}$ are functions of the coordinates only (henceforth all superscripted quantities will be functions of the coordinates only).

Since the relation between the stresses and strains is everywhere described by the single analytic relation (1), it is only natural to seek the solution of the problem for the stresses in the form

$$\sigma_{ij} = \sum_{k=1} \sigma_{ij}^{(k)} \lambda^k\ . \tag{6}$$

The hydrostatic pressure σ is given by the relation

$$\sigma = \frac{1}{3}\sigma_{ii} = \frac{1}{3}\sum \sigma_{ii}^{(k)} \lambda^k\ .$$

Hence the deviator

$$S_{ij} = \sigma_{ij} - \sigma\delta_{ij} = \sum_{k=1} S_{ij}^{(k)} \lambda^k \quad (\delta_{ij} \text{ — Kronecker delta})\ , \tag{7}$$

$$S_{ij}^{(k)} = \sigma_{ij}^{(k)} - \sigma^{(k)}\delta_{ij}\ , \qquad \sigma^{(k)} = \frac{1}{3}\sigma_{ii}^{(k)} \tag{8}$$

and

$$S_{ij}S_{ij} = \sum_{k=1} S_{ij}^{(k)}\lambda^k \sum_{n=1} S_{ij}^{(n)}\lambda^n = \sum_{n=2} a_n\lambda^n\ , \quad a_n = \sum_{m=1}^{m=n-1} S_{ij}^{(m)} S_{ij}^{(n-m)}\ . \tag{9}$$

The following series is obtained for $F(T)$:

$$F(T)=\sum_{n=2}\frac{1}{n}\chi_n\lambda^n, \qquad \chi_n=n\sum_{\alpha=1}A_{2\alpha}\sum_{k_1+\ldots+k_\alpha=n}a_{k_1}\cdots a_{k_\alpha}\ . \tag{10}$$

Clearly, χ_n is defined in terms of $S_{ij}^{(k)}$ with $k < n$. Substituting equations (7) and (10) in equation (1) and then integrating, taking into account the zero initial conditions, we get:

$$2G\varepsilon_{ij}=\sum_{k=1}S_{ij}^{(k)}\lambda^k+\sum_{m=2}\left[\frac{1}{m+2}\sum_{n=2}^{n=m}\chi_n S_{ij}^{(m-n+1)}\right]\lambda^{m+1}\ . \tag{11}$$

This can be represented in the form

$$2G\,\varepsilon_{ij}=S_{ij}^{(1)}\lambda+S_{ij}^{(2)}\lambda^2+\sum_{k=3}(S_{ij}^{(k)}+R_{ij}^{(k)})\lambda^k\ , \tag{12}$$

where

$$R_{ij}^{(k)}=\frac{1}{k}\sum_{n=2}^{n=k-1}\chi_n S_{ij}^{(k-n)}, \qquad R_{ij}^{(1)}=R_{ij}^{(2)}=0\ . \tag{13}$$

We shall introduce the new tensors

$$\sigma_{ij}^{(k)*}=\sigma_{ij}^{(k)}+N_{ij}^{(k)}. \qquad 2G\varepsilon_{ij}^{(k)*}=S_{ij}^{(k)}+R_{ij}^{(k)} \tag{14}$$

$$N_{ij}^{(k)}=\frac{1}{k}\sum_{n=2}^{n=k-1}\chi_n\sigma_{ij}^{(k-n)}, \qquad N_{ij}^{(1)}=N_{ij}^{(2)}=0\ . \tag{15}$$

Since $R_{ij}^{(k)}=N_{ij}^{(k)}-(1/3)N_{ii}^{(k)}\delta_{ij}$,

$$2G\varepsilon_{ij}^{(k)*}=S_{ij}^{(k)*}=\sigma_{ij}^{(k)*}-\frac{1}{3}\sigma^{(k)}\sigma_{ij}^{*}\ , \tag{16}$$

i.e., the tensors $\varepsilon_{ij}^{(k)*}$ and $\sigma_{ij}^{(k)*}$ are linked by Hooke's Law for an incompressible material.

In the new notation Eq. (12) assumes the form

$$\varepsilon_{ij}=\sum_{k=1}\varepsilon_{ij}^{(k)*}\lambda^k. \tag{17}$$

By virtue of the fact that the compatibility equations must be fulfilled for any value of λ, they must be fulfilled for any $\varepsilon_{ij}^{(k)*}$. Analogously, the equilibrium equations must be fulfilled for any $\sigma_{ij}^{(k)}$

$$\frac{\partial\sigma_{ij}^{(k)}}{\partial x_j}+V_i^{(k)}=0\ . \tag{18}$$

Introducing $\sigma_{ij}^{(k)*}$, from Eq. (14) we get:

$$\frac{\partial\sigma_{ij}^{(k)*}}{\partial x_j}+V_i^{(k)}=\frac{\partial N_{ij}^{(k)}}{\partial x_j}\ . \tag{19}$$

Then at the surface of the body we have:

$$\sigma_{ij}^{(k)}l_j=F_i^{(k)}, \tag{20}$$

where l_j are the direction cosines of the exterior normal. Introducing Eq. (14), we get:

$$\sigma_{ij}^{(k)*}l_j=F_i^{(k)}+N_{ij}^{(k)}l_j\ . \tag{21}$$

Clearly, it is necessary to solve the elasticity problem for the tensors $\sigma_{ij}^{(k)*}$ and $\varepsilon_{ij}^{(k)*}$ in relation to the surface and mass forces

$$F_i^{(k)*}=F_i^{(k)}+N_{ij}^{(k)}l_j, \quad V_i^{(k)*}=V_i^{(k)}-\frac{\partial N_{ij}^{(k)}}{\partial x_j}\ , \tag{22}$$

respectively. Note that, by virtue of the realizability of equations (18) and (20), for any $\sigma_{ij}^{(m)}$

$$\frac{\partial N_{ij}^{(k)}}{\partial x_j}=\frac{1}{k}\sum_{n=2}^{n=k-1}\left(\sigma_{ij}^{(k-n)}\frac{\partial\chi_n}{\partial x_j}-\chi_n V_i^{(k-n)}\right)\ , \tag{23}$$

$$N_{ij}^{(k)} l_j = \frac{1}{k} \sum_{n=2}^{n=k-1} \chi_n F_i^{(k-n)} .$$

As it is easy to see, the tensors $N_{ij}^{(k)}$ are found in terms of solutions with an index less than k. Hence, the method of successive determination of $\sigma_{ij}^{(k)*}$ and $\varepsilon_{ij}^{(k)*}$, and hence of $\sigma_{ij}^{(k)}$, is possible. Since $N_{ij}^{(1)} = N_{ij}^{(2)} = 0$, for $\sigma_{ij}^{(1)*} = \sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2*)} = \sigma_{ij}^{(2)}$ we have a problem of elasticity with $F_i^{(k)*} = F_i^{(k)}$ and $V_i^{(k)*} = V_i^{(k)}$. We then find $N_{ij}^{(3)} = 1/3\, \chi_2 \sigma_{ij}^{(1)}$ and determine $\sigma_{ij}^{(3)*}$ and $\varepsilon_{ij}^{(3)*}$, solving the elasticity problem for the external forces given by (22). Now, from Eq. (14)

$$\sigma_{ij}^{(3)} = \sigma_{ij}^{(3)*} - N_{ij}^{(3)} ,$$

and, knowing $\sigma_{ij}^{(3)}$, we find

$$N_{ij}^{(4)} = \frac{1}{4} \left(\chi_2 \sigma_{ij}^{(2)} + \chi_3 \sigma_{ij}^{(1)}\right) \text{and so on.}$$

Clearly, $N_{ij}^{(m)}$ is expressed in terms of an elastic solution with an index m−2 or lower.

Hence, the problem of complex loading reduces to the successive solution of elasticity problems with certain fictitious external forces. As usual, the computations are carried on until the difference between the internal fields for two successive elastic problems is sufficiently small. Clearly, in contradistinction to the step method, it is not necessary to repeat the entire computation to refine the solution.

In practical computations, it is possible to limit series (3) to one or two terms, if we take into consideration the scatter of experimental data for determining the function F(T). Retaining only one term gives an approximation to the curve for simple tension in the form of a cubic parabola. Then $X_n = na_n$, and the computations are much simplified.

REFERENCES

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